Dialectica logical principles via free categorical constructions

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Introduction: Dialectica interpretation

Gödel's Dialectica Interpretation: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type, called system T.

Idea: translate every formula A of HA to $A^D = \exists x \forall y A_D$, where A_D is quantifier-free.

Application: if HA proves A, then system T proves $A_D(t, y)$, where y is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing y).

Goal: to be as constructive as possible, while being able to interpret all of classical arithmetic.

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes*, Dialectica, 12(3-4):280–287.

Introduction: Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(\psi \to \phi)^D$:

$$(\psi \to \phi)^D = \exists V, X. \forall u, y. (\psi_D(u, X(u, y)) \to \phi_D(V(u), y)).$$

This involves three logical principles: a form of the **Principle of Independence of Premise** (IP), a generalisation of **Markov's Principle** (MP), and the **axiom of choice** (AC).

Intuition: given a witness u for the hypothesis ψ_D, there exists a function V assigning a witness $V(u)$ of ϕ_D to every witness u of ψ_D. Moreover, this assignment has to be such that from a counterexample y of the conclusion ϕ_D we should be able to find a counterexample $X(u, y)$ to the hypothesis ψ_D.

Gödel, Feferman, et al (1986), *Kurt Gödel: Collected Works: Volume II:*, Oxford University Press.

Introduction: Dialectica interpretation in category theory

Dialectica category: given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathsf{C})$, the objects of which have the form (U, X, α) where α is a subobject of $U \times X$ in C; such an object is thought of as the formula

$$\exists u \forall x \alpha(u, x).$$

An arrow from $\exists u \forall x \alpha(u, x)$ to $\exists y \forall v \beta(y, v)$ can be thought of as a pair (f_0, f_1) of terms, subject to the condition

$$\alpha(u, f_1(u, v)) \vdash \beta(f_0(u), v).$$

Generalization: the construction introduced by de Paiva has been generalized for arbitrary fibrations.

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de Paiva (1991), *The Dialectica categories*, PhD Thesis.

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration*, 46th International Symposium on Mathematical Foundations of Computer Science, 87:1-87:16

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Dialectica logical principles categorically

- In what way does the construction of these Dialectica categories (or fibrations) capture the essential ingredients of Gödel's original translation, namely (IP), (MP) and (AC)?
- Can they be described in more conceptual terms, for example in terms of universal properties?

Doctrines

Definition

A **doctrine** is just a functor:

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

where the category $\mathcal{C}$ has finite products and **Pos** is the category of posets.

Definition

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is ***existential*** (resp. ***universal***) if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor:

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the ***Beck-Chevalley condition***.

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an existential doctrine and let A be an object of $\mathcal{C}$. A predicate α of the fibre $P(A)$ is said to be an ***existential splitting*** if it satisfies the following universal property: for every projection $A \times B \xrightarrow{\pi_A} A$ of $\mathcal{C}$ and every predicate $\beta \in P(A \times B)$ such that $\alpha \le \exists_{\pi_A}(\beta)$, there exists an arrow $A \xrightarrow{g} B$ such that:

$$\alpha \le P_{\langle 1_A, g\rangle}(\beta).$$

Existential splittings stable under re-indexing are called *existential-free elements*. Thus we introduce the following definition:

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an existential doctrine and let I be an object of $\mathcal{C}$. A predicate α of the fibre $P(I)$ is said to be ***existential-free*** if $P_f(\alpha)$ is an existential splitting for every morphism $A \xrightarrow{f} I$.

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a universal doctrine and let A be an object of $\mathcal{C}$. A predicate α of the fibre $P(A)$ is said to be a ***universal splitting*** if it satisfies the following universal property: for every projection $A \times B \xrightarrow{\pi_A} A$ of $\mathcal{C}$ and every predicate $\beta \in P(A \times B)$ such that $\forall_{\pi_A}(\beta) \leq \alpha$, there exists an arrow $A \xrightarrow{g} B$ such that:

$$P_{\langle 1_A, g\rangle}(\beta) \leq \alpha.$$

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a universal doctrine and let I be an object of $\mathcal{C}$. A predicate α of the fibre $P(I)$ is said to be ***universal-free*** if $P_f(\alpha)$ is a universal splitting for every morphism $A \xrightarrow{f} I$.

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a doctrine. If P is existential, we say that P has ***enough existential-free predicates*** if, for every object I of $\mathcal{C}$ and every predicate $\alpha \in PI$, there exist an object A and an existential-free object β in $P(I \times A)$ such that $\alpha = \exists_{\pi_I}\beta$.

Analogously, if P is universal, we say that P has ***enough universal-free predicates*** if, for every object I of $\mathcal{C}$ and every predicate $\alpha \in PI$, there exist an object A and a universal-free object β in $P(I \times A)$ such that $\alpha = \forall_{\pi_I}\beta$.

Definition

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is called a ***Gödel doctrine*** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine P is existential and universal;
3. the doctrine P has enough existential-free predicates;
4. the existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_{\pi}(\alpha)$ is existential-free for every projection π from A;
5. the sub-doctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of the existential-free predicates of P has enough universal-free predicates.

An element α of a fibre $P(A)$ of a Gödel doctrine P that is both an existential-free predicate and a universal-free predicate in the sub-doctrine P' of existential-free elements of P is called a **quantifier-free predicate** of P.

Theorem

Let $P : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ *be a Gödel doctrine, and let* α *be an element of* $P(A)$*. Then there exists a quantifier-free predicate* α_D *of* $P(I \times U \times X)$ *such that:*

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U.\forall x : X.\alpha_D(i, u, x).$$

Theorem

Let $P : \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ *be a Gödel doctrine. Then for every* $\psi_D \in P(I \times U \times X)$ *and* $\phi_D \in P(I \times V \times Y)$ *quantifier-free predicates of* P *we have that:*

$$i : I \mid \exists u.\forall x.\psi_D(i, u, x) \vdash \exists v.\forall y.\phi_D(i, v, y)$$

if and only if there exists $I \times U \xrightarrow{f_0} V$ *and* $I \times U \times Y \xrightarrow{f_1} X$ *such that:*

$$u : U, y : Y, i : I \mid \psi_D(i, u, f_1(i, u, y)) \vdash \phi_D(i, f_0(i, u), y).$$

Theorem

Every Gödel doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *validates the* ***Skolemisation principle****, that is:*

$$a_1 : A_1 \mid \forall a_2.\exists b.\alpha(a_1, a_2, b) \dashv\vdash \exists f.\forall a_2.\alpha(a_1, a_2, fa_2)$$

where $f : B^{A_2}$ *and* fa_2 *denote the evaluation of* f *on* a_2*, whenever* $\alpha(a_1, a_2, b)$ *is a predicate in the context* $A_1 \times A_2 \times B$*.*

Theorem

Every Gödel doctrine P *is equivalent to the Dialectica completion* $\mathfrak{Dial}(P')$ *of the full subdoctrine* P' *of* P *consisting of the quantifier-free predicates of* P*.*

Gödel hyperdoctrine

A ***hyperdoctrine*** is a functor:

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$$

from a cartesian closed category $\mathcal{C}$ to the category of Heyting algebras **Hey** satisfying some further conditions: for every arrow $A \xrightarrow{f} B$ in $\mathcal{C}$, the homomorphism $P_f\colon P(B) \longrightarrow P(A)$ of Heyting algebras, where P_f denotes the action of the functor P on the arrow f, has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ satisfying the Beck-Chevalley conditions.

Definition

A hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$ is said a ***Gödel hyperdoctrine*** when P is a Gödel doctrine.

Theorem

Every Gödel hyperdoctrine $P \colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *satisfies the* ***Rule of Independence of Premise****, i.e. whenever* $\beta \in P(A \times B)$ *and* $\alpha \in P(A)$ *is a existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash \alpha(a) \to \exists b.\beta(a, b) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a) \to \beta(a, b)).$$

Theorem

Every Gödel hyperdoctrine $P \colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *satisfies the following* ***Modified Markov's Rule****, i.e. whenever* $\beta_D \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash (\forall b.\alpha(a, b)) \to \beta_D(a) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a, b) \to \beta_D(a)).$$

Corollary

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *such that* $\bot$ *is a quantifier-free predicate satisfies* ***Markov's Rule****, i.e. for every quantifier-free element* $\alpha_D \in P(A \times B)$ *it is the case that:*

$$b : B \mid \top \vdash \neg \forall a.\alpha_D(a, b) \text{ implies that } b : B \mid \top \vdash \exists a.\neg \alpha_D(a, b).$$

Corollary

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *such that* $\top$ *is existential-free satisfies the* ***Rule of Choice****, that is, whenever:*

$$a : A \mid \top \vdash \exists b.\alpha(a, b)$$

for some existential-free predicate $\alpha \in P(A \times B)$*, then it is the case that:*

$$a : A \mid \top \vdash \alpha(a, g(a))$$