Generalized existential completions, choice rules, exact completions*

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The process of completing a category with quotients by producing an exact or regular category introduced in [8, 9, 7] has been widely studied in the literature of category theory with applications both to mathematics and to computer science, see [1, 2, 6].

In the works [10, 11, 12] the notions of exact completion of a weakly cartesian finite product category and of a regular category turns out to be the base categories of two different instances of a completion with exact quotients relative to a Lawvere elementary existential doctrine which is the fundamental structure underlying the the tripos-to-topos construction in [3, 13].

Our first aim is to show that these various notions of quotient completions can be further extended by freely adding exact, regular, elementary quotients to a primary doctrine P equipped with a class of morphisms Λ in its base category where Λ is closed under composition, identities.

The key result which allows this characterization, is the introduction of notion of Λ-existential doctrine and Λ-existential completion. In particular, the characterization of the free algebras

Theorem. *Let* $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ *be a* Λ*-existential doctrine. Then* P *is an instance of* Λ*- existential completion if and only if*

1. *P satisfies Λ-(RC);*
2. *P has enough-Λ-existential-free objects;*
3. *for every Λ-existential-free object α and β of $P(A)$, then $\alpha \wedge \beta$ is Λ-existential-free object.*

Then, we show how, using the notion of generalized existential completion, we can provide a characterization of those elementary and existential doctrines $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ whose exact completion $\mathcal{T}_P$ is an instance of the $(-)_{\text{ex}\,/\,\text{lex}}$ completion.

Theorem. *Let* $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then the regular completion* $\mathcal{E}f_{P_{cx}}$ *of the doctrine* P *is an instance of the* $(-)_{\text{reg}\,/\,\text{lex}}$ *completion if and only if* $\mathcal{E}f_{P_{cx}} \equiv \mathcal{E}f_{P'_{cx}}$*, where* P' *is an instance of* Ω*-existential completion.*

Theorem. *Let* $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then the exact completion* $\mathcal{T}_P$ *of the doctrine* P *is an instance of the* $(-)_{\text{ex}\,/\,\text{lex}}$ *completion if and only if* $\mathcal{T}_P \equiv \mathcal{T}_{P'}$*, where* P' *is an instance of* Ω*-existential completion.*

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