Gödel doctrines and Dialectica logical principles

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Introduction: Dialectica interpretation

Gödel's Dialectica Interpretation: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type, called system T.

Idea: translate every formula A of HA to $A^D = \exists x \forall y A_D$, where A_D is quantifier-free.

Application: if HA proves A, then system T proves $A_D(t, y)$, where y is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing y).

Goal: to be as constructive as possible, while being able to interpret all of classical arithmetic.

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes*, Dialectica, 12(3-4):280–287.

Introduction: Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(\psi \to \phi)^D$:

$$(\psi \to \phi)^D = \exists f_0, f_1. \forall u, y. (\psi_D(u, f_1(u, y)) \to \phi_D(f_0(u), y)).$$

This involves three logical principles: a form of the **Principle of Independence of Premise** (IP), a generalisation of **Markov's Principle** (MP), and the **axiom of choice** (AC).

Intuition: given a witness u for the hypothesis ψ_D, there exists a function f_0 assigning a witness $f_0(u)$ of ϕ_D to every witness u of ψ_D. Moreover, this assignment has to be such that from a counterexample y of the conclusion ϕ_D we should be able to find a counterexample $f_1(u, y)$ to the hypothesis ψ_D.

Gödel, Feferman, et al (1986), *Kurt Gödel: Collected Works: Volume II:*, Oxford University Press.

Introduction: Dialectica interpretation in category theory

Dialectica category: given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathrm{C})$, the objects of which have the form (U, X, ψ) where ψ is a subobject of $U \times X$ in C; such an object is thought of as the formula

$$\exists u \forall x \psi(u, x).$$

An arrow from $\exists u \forall x \psi(u, x)$ to $\exists v \forall y \phi(v, y)$ can be thought of as a pair (f_0, f_1) of terms, subject to the condition

$$\psi(u, f_1(u, y)) \vdash \phi(f_0(u), y).$$

The definition of morphism is motivated by the way the dialectica interpretation acts on implicational formulae.

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de Paiva (1991), *The Dialectica categories*, PhD Thesis.

Introduction: Dialectica interpretation

Generalization: the construction introduced by de Paiva has been generalized for arbitrary fibrations.
Dialectica pseudo-monad: given a fibration p, one can construct the Dialectica fibration $\mathfrak{Dial}(p)$. Moreover, under the assumption that the base category of p is cartesian closed, this construction is monadic.

In this talk we will use a presentation of the Dialectica construction in terms of **Lawvere's doctrines**.

Hyland (2002), *Proof theory in the abstract*, Annals of Pure and Applied Logic, 114(1):43 - 78

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration*, 46th International Symposium on Mathematical Foundations of Computer Science, 87:1-87:16

Our contributions

- Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?
- When such doctrine P' exists, how can we find it?

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration*, 46th International Symposium on Mathematical Foundations of Computer Science, 87:1-87:16

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  Such a P' exists precisely when P is a Gödel doctrine
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Applications: we can easily provide an answer to the following questions

- In what way does the construction of these Dialectica categories (or fibrations) capture the essential ingredients of Gödel's original translation, namely (IP), (MP) and (AC)?
- Can they be described in more conceptual terms, for example in terms of universal properties?

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Doctrines

Definition

A **doctrine** is just a functor:

$$P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

where the category $\mathcal{C}$ has finite products and **Pos** is the category of posets.

Syntactic intuition

Doctrines can be seen as the generalisation of the so-called **Lindenbaum-Tarski algebra**: given a first order theory $\mathcal{T}$ in a first order language $\mathcal{L}$, one can consider the functor

$$\mathcal{LT} : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

whose base category $\mathcal{V}$ is the **syntactic** category of $\mathcal{L}$, i.e. the objects of $\mathcal{V}$ are finite lists $\overrightarrow{x} := (x_1, \dots, x_n)$ of variables and morphisms are lists of substitutions, while the elements of $\mathcal{LT}(\overrightarrow{x})$ are given by equivalence classes of well-formed formulae in the context $\overrightarrow{x}$, and order is given by the provable consequences with respect to the fixed theory $\mathcal{T}$.

Semantic intuition

Semantically, a doctrine is essentially a generalisation of the contravariant **power-set functor** on the category of sets:

$$\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

sending any set-theoretic arrow $A \xrightarrow{f} B$ to the inverse image functor $\mathcal{P}B \xrightarrow{\mathcal{P}f = f^{-1}} \mathcal{P}A$.

Existential and Universal doctrines

Definition

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is **existential** if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$, and these satisfy the ***Beck-Chevalley condition***.

Definition

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is **universal** if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a right adjoint $\forall_{\pi_i}$, and these satisfy the ***Beck-Chevalley condition***.

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an existential doctrine and let A be an object of $\mathcal{C}$. A predicate α of the fibre $P(A)$ is said to be an ***existential splitting*** if it satisfies the following universal property: for every projection $A \times B \xrightarrow{\pi_A} A$ of $\mathcal{C}$ and every predicate $\beta \in P(A \times B)$ such that $\alpha \leq \exists_{\pi_A}(\beta)$, there exists an arrow $A \xrightarrow{g} B$ such that:

$$\alpha \leq P_{\langle 1_A, g\rangle}(\beta).$$

Existential splittings stable under re-indexing are called *existential-free elements*. Thus we introduce the following definition:

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an existential doctrine and let I be an object of $\mathcal{C}$. A predicate α of the fibre $P(I)$ is said to be ***existential-free*** if $P_f(\alpha)$ is an existential splitting for every morphism $A \xrightarrow{f} I$.

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a doctrine. If P is existential, we say that P has ***enough existential-free predicates*** if, for every object I of $\mathcal{C}$ and every predicate $\alpha \in PI$, there exist an object A and an existential-free object β in $P(I \times A)$ such that $\alpha = \exists_{\pi_I}\beta$.

Analogously, if P is universal, we can introduce the notions of universal splitting and universal-free elements. We say that P has ***enough universal-free predicates*** if, for every object I of $\mathcal{C}$ and every predicate $\alpha \in PI$, there exist an object A and a universal-free object β in $P(I \times A)$ such that $\alpha = \forall_{\pi_I}\beta$.

Maietti and Trotta (2021), *Generalized existential completions and their regular and exact completions*, preprint

Notation. From now on, we shall employ the logical language provided by the **internal language** of a doctrine and write:

$$a_1 : A_1, \dots, a_n : A_n \mid \phi(a_1, \dots, a_n) \vdash \psi(a_1, \dots, a_n)$$

instead of:

$$\phi \leq \psi$$

in the fibre $P(A_1 \times \cdots \times A_n)$. Similarly, we write:

$$a : A \mid \phi(a) \vdash \exists b : B. \psi(a, b) \text{ and } a : A \mid \phi(a) \vdash \forall b : B. \psi(a, b)$$

in place of:

$$\phi \leq \exists_{\pi_A} \psi \text{ and } \phi \leq \forall_{\pi_A} \psi$$

in the fibre $P(A)$. Also, we write $a : A \mid \phi \dashv\vdash \psi$ to abbreviate $a : A \mid \phi \vdash \psi$ and $a : A \mid \psi \vdash \phi$.

Definition

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is called a ***Gödel doctrine*** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine P is existential and universal;
3. the doctrine P has enough existential-free predicates;
4. the existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_{\pi}(\alpha)$ is existential-free for every projection π from A;
5. the sub-doctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of the existential-free predicates of P has enough universal-free predicates.

An element α of a fibre $P(A)$ of a Gödel doctrine P that is both an existential-free predicate and a universal-free predicate in the sub-doctrine P' of existential-free elements of P is called a **quantifier-free predicate** of P.

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *be a Gödel doctrine, and let* α *be an element of* $P(I)$*. Then there exists a quantifier-free predicate* α_D *of* $P(I \times U \times X)$ *such that:*

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U. \forall x : X. \alpha_D(i, u, x).$$

Theorem

Every Gödel doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *validates the* ***Skolemisation principle****, that is:*

$$a_1 : A_1 \mid \forall a_2. \exists b. \alpha(a_1, a_2, b) \dashv\vdash \exists f. \forall a_2. \alpha(a_1, a_2, fa_2)$$

where $f : B^{A_2}$ *and* fa_2 *denote the evaluation of* f *on* a_2*, whenever* $\alpha(a_1, a_2, b)$ *is a predicate in the context* $A_1 \times A_2 \times B$*.*

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *be a Gödel doctrine. Then for every* $\psi_D \in P(I \times U \times X)$ *and* $\phi_D \in P(I \times V \times Y)$ *quantifier-free predicates of P we have that:*

$$i : I \mid \exists u.\forall x.\psi_D(i, u, x) \vdash \exists v.\forall y.\phi_D(i, v, y)$$

if and only if there exists $I \times U \xrightarrow{f_0} V$ *and* $I \times U \times Y \xrightarrow{f_1} X$ *such that:*

$$u : U, y : Y, i : I \mid \psi_D(i, u, f_1(i, u, y)) \vdash \phi_D(i, f_0(i, u), y).$$

Dialectica doctrine

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a doctrine whose base category C is cartesian closed. We define the **dialectica doctrine** $\mathfrak{Dial}(P)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ the functor sending an object I into the poset $\mathfrak{Dial}(P)(I)$ defined as follows:

- **objects** are quadruples (I, X, U, α) where I, X and U are objects of the base category $\mathcal{C}$ and $\alpha \in P(I \times X \times U)$;
- **partial order:** we stipulate that $(I, U, X, \alpha) \leq (I, V, Y, \beta)$ if there exists a pair (f_0, f_1), where $I \times U \xrightarrow{f_0} V$ and $I \times U \times Y \xrightarrow{f_1} X$ are morphisms of $\mathcal{C}$ such that:

$$\alpha(i, u, f_1(i, u, y)) \leq \beta(i, f_0(i, u), y).$$

Gödel doctrine iff Dialectica doctrine

Theorem

Let $P\colon \mathsf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *be an existential and universal doctrine whose base category* C *is cartesian closed.*
Then P *is equivalent to the Dialectica completion* $\mathfrak{Dial}(P')$ *of a full subdoctrine* P' *of* P *if and only if* P *is a Gödel doctrine. In this case,* P' *consists of the quantifier-free predicates of* P*.*

Sketch of the proof

The original Dialectica construction $\mathfrak{Dial}$ can be seen as the composition of two free constructions $\mathfrak{Sum}$ and $\mathfrak{Prod}$, which are the existential and the universal completions, respectively.

Lemma

There is an isomorphism of doctrines, natural in P:

$$\mathfrak{Dial}(P) \cong \mathfrak{Sum}(\mathfrak{Prod}(P)).$$

These completions are fully dual, in particular $\mathfrak{Prod}(\mathrm{p}) \cong \mathfrak{Sum}(\mathrm{p}^{\mathrm{op}})^{\mathrm{op}}$, so we only need to study one and can then deduce results for the other construction.

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Sketch of the proof

Theorem

An existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ *is an instance of the existential completion if and only if it has enough existential-free objects. Moreover, in this case* $P \cong \mathfrak{Sum}(P')$ *where* P' *is the subdoctrine of existential-free elements of* P*.*

Theorem

An universal doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ *is an instance of the universal completion if and only if it has enough universal-free objects. Moreover, in this case* $P \cong \mathfrak{Prod}(P')$ *where* P' *is the subdoctrine of universal-free elements of* P*.*

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Gödel hyperdoctrine

A ***hyperdoctrine*** is a functor:

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$$

from a cartesian closed category $\mathcal{C}$ to the category of Heyting algebras **Hey** satisfying some further conditions: for every arrow $A \xrightarrow{f} B$ in $\mathcal{C}$, the homomorphism $P_f\colon P(B) \longrightarrow P(A)$ of Heyting algebras, where P_f denotes the action of the functor P on the arrow f, has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ satisfying the Beck-Chevalley conditions.

Definition

A hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$ is said a ***Gödel hyperdoctrine*** when P is a Gödel doctrine.

Theorem

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *satisfies the* ***Rule of Independence of Premise****, i.e. whenever* $\beta \in P(A \times B)$ *and* $\alpha \in P(A)$ *is a existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash \alpha(a) \to \exists b.\beta(a, b) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a) \to \beta(a, b)).$$

Theorem

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *satisfies the following* ***Modified Markov's Rule****, i.e. whenever* $\beta_D \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash (\forall b.\alpha(a, b)) \to \beta_D(a) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a, b) \to \beta_D(a)).$$

Corollary

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *such that* $\bot$ *is a quantifier-free predicate satisfies* ***Markov's Rule****, i.e. for every quantifier-free element* $\alpha_D \in P(A \times B)$ *it is the case that:*

$$b : B \mid \top \vdash \neg\forall a.\alpha_D(a, b) \text{ implies that } b : B \mid \top \vdash \exists a.\neg\alpha_D(a, b).$$

Corollary

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Hey}$ *such that* $\top$ *is existential-free satisfies the* ***Rule of Choice****, that is, whenever:*

$$a : A \mid \top \vdash \exists b.\alpha(a, b)$$

for some existential-free predicate $\alpha \in P(A \times B)$*, then it is the case that:*

$$a : A \mid \top \vdash \alpha(a, g(a))$$

Future work

- Employing the notion of Gödel doctrine as ”bridge” to compare categorically Hilbert’s epsilon-calculus and Dialectica interpretation;
- explore connections with the notion of *softness*;
- combine this notion with the results of our work with Milly Maietti, where we show that the tripos-to-topos of a tripos with enough-existential-free elements is an instance of the ex/lex completion.