The Hilbert ε-operator and existence property in categorical logic

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Introduction

Let **Sg** be a first order many-sorted signature and let Th be a theory. We define a functor

$$\mathrm{LT}\colon \mathcal{C}_{\mathsf{Th}}^{op} \longrightarrow \textbf{InfSL}$$

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- a **morphisms** from $[x_1 : \sigma_1, \dots, x_n : \sigma_n]$ to $[y_1 : \tau_1, \dots, y_m : \tau_m]$ is an equivalence class $\gamma := [t_1 : \tau_1, \dots, t_m : \tau_m]$ where

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  are well-formed terms-in-context;
- the **composition** of two morphisms $\gamma\colon \Gamma \longrightarrow \Gamma'$ and $\gamma'\colon \Gamma' \longrightarrow \Gamma''$ is given by the substitution

  $$[s_1[\vec{t}/\vec{y}], \dots, s_k[\vec{t}/\vec{y}]]\colon \Gamma \longrightarrow \Gamma'' .$$

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We say that $\psi \leq \phi$ where $\phi, \psi \in \mathrm{LT}(\Gamma)$ if $\psi \vdash_{\mathsf{Th}} \phi$, and then we quotient in the usual way to obtain a partial order on $\mathrm{LT}(\Gamma)$. Given a morphism of $\mathcal{C}_{\mathsf{Th}}$

$$\gamma = [t_1 : \tau_1, \dots, t_m : \tau_m]\colon \Gamma \longrightarrow \Gamma'$$

the functor LT_{γ} acts as the substitution $\mathrm{LT}_{\gamma}(\psi(y_1, \dots, y_m)) = \psi[\vec{t}/\vec{y}]$.

Doctrines

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Definition

A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential if, for every A_1, A_2 in $\mathcal{C}$, for any projection $pr_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor

$$P_{pr_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$

has a left adjoint $\exists_{pr_i}$, and these satisfy BC and FR.

Example

Set-theoretic doctrine. Let **Set** be the category of sets and functions.

$$\mathrm{S}\colon \ \mathbf{Set}^{op} \longrightarrow \mathbf{InfSL}\ .$$

For every set A, $\mathrm{S}(A)$ is the poset category of subsets of the set A whose morphisms are inclusions, and for every function $f\colon A \longrightarrow B$ the functor $\mathrm{S}_f\colon \mathrm{S}(B) \longrightarrow \mathrm{S}(A)$ acts as the inverse image $f^{-1}(U)$ for every subset U of B.

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- P **satisfies the Rule of Choice** (RC) if for every $\phi \in P(A \times B)$ such that
$$a : A \mid \top \vdash \exists b : B\ \phi(a,b)$$
there exists an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that
$$a : A \mid \top \vdash \phi(a, f(a)).$$

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- P is **equipped with ϵ-operators** if for every object B and A in $\mathcal{C}$ and any α in $P(A \times B)$ there exists an arrow $\epsilon_\alpha\colon A \longrightarrow B$ such that

$$a : A \mid \exists b : B\ \alpha(a,b) \dashv\vdash \alpha(a, \epsilon_\alpha(a))$$

holds in $P(A)$.

Choice Principles

- an elementary existential doctrine P **satisfies the Rule of Unique Choice** (RUC) if for every entire functional $\phi \in P(A \times B)$, i.e
  1. $a : A \mid \top \vdash \exists b : B\ \phi(a, b)$
  2. $a : A, b : B, b' : B \mid \phi(a, b) \wedge \phi(a, b') \vdash b =_B b'$

  there exists an arrow $f : A \longrightarrow B$ in $\mathcal{C}$ such that

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We define Ef_P the **category of entire functional relations** of P: objects are those of $\mathcal{C}$ and an arrow $\phi\colon A \longrightarrow B$ is an entire functional relation from A to B.

The Existential Completion

Let $P\colon \mathcal{C}^{op} \longrightarrow \textbf{InfSL}$ be a primary doctrine and let $\mathcal{A} \subset \mathcal{C}_1$ be the class of projections. For every object A of $\mathcal{C}$ consider we define $P^e(A)$ the following poset:

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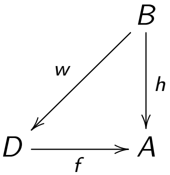
- the objects are pairs $(\ B \xrightarrow{g\in\mathcal{A}} A\ ,\ \alpha \in PB)$;
- $(\ B \xrightarrow{h\in\mathcal{A}} A\ ,\ \alpha \in PB) \leq (\ D \xrightarrow{f\in\mathcal{A}} A\ ,\ \gamma \in PD)$ if there exists $w\colon B \longrightarrow D$ such that



commutes and $\alpha \leq P_w(\gamma)$.

Theorem (RC)

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine. Then the doctrine* $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *satisfies the Rule of Choice.*

Theorem (GEP)

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine, and consider the doctrine* $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *. If*

$$a : A \mid \alpha(a) \vdash \exists b : B\ \beta(a, b)$$

then there exists an arrow $t\colon A \longrightarrow B$ *such that*

$$a : A \mid \alpha(a) \vdash \beta(a, t(a)).$$

The syntactic doctrine

$$LT_{\mathcal{L}_{=,\exists}} : \mathcal{C}^{op}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

is the existential completion of the syntactic doctrine

$$LT_{\mathcal{L}_{=}} : \mathcal{C}^{op}_{\mathcal{L}_{=}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{L}_{=,\exists}$ is the Regular fragment of first order intuitionistic logic, and $\mathcal{L}_{=}$ is the Horn fragment.
Therefore the Regular fragment of first order intuitionistic logic satisfies RC and GEP.

For every existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ there is a canonical arrow $\varepsilon_P\colon P^e \longrightarrow P$ of existential doctrine which acts on $P^e(A)$ as

$$(\, A\times B \xrightarrow{pr_A} A \,, \alpha\in P(A\times B)) \mapsto \exists_{pr_A}(\alpha).$$

If the doctrine is elementary and existential ε_P induces a functor

$$\overline{\varepsilon_P}\colon \mathrm{Ef}_{P^e} \longrightarrow \mathrm{Ef}_P$$

on the categories of entire functional relations.

Theorem

An existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is equipped with* ϵ*-operators if and only if* $\varepsilon_P\colon P^e \longrightarrow P$ *is an isomorphism.*

Theorem

An existential elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *satisfies RUC if and only if the functor* $\overline{\varepsilon_P}\colon \mathrm{Ef}_{P^e} \longrightarrow \mathrm{Ef}_P$ *is full.*