The Gödel fibration

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Introduction: Dialectica interpretation

Gödel's Dialectica Interpretation: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type, called system T.

Idea: translate every formula A of HA to $A^D = \exists x \forall y A_D$, where A_D is quantifier-free.

Application: if HA proves A, then system T proves $A_D(t, y)$, where y is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing y).

Goal: to be as constructive as possible, while being able to interpret all of classical arithmetic.

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes*, Dialectica, 12(3-4):280–287.

Introduction: Dialectica interpretation

Dialectica category: given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathsf{C})$, the objects of which have the form (X, U, α) where α is a subobject of $X \times U$ in C; such an object is thought of as the formula

$$\exists x \forall u \alpha(x, u).$$

An arrow from $\exists x \forall u \alpha(x, u)$ to $\exists y \forall v \beta(y, v)$ can be thought of as a pair (f, f_0) of terms, subject to the condition

$$\alpha(x, f_0(x, v)) \vdash \beta(f(x), v).$$

The definition of morphism is motivated by the way the dialectica interpretation acts on implicational formulae.

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de Paiva (1991), *The Dialectica categories*, PhD Thesis.

Introduction: Dialectica interpretation

Dialectica pseudo-monad: given a fibration p, one can construct the Dialectica fibration $\mathfrak{Dial}(\mathrm{p})$. Moreover, under the assumption that the base category of p is cartesian closed, this construction is monadic.

Hyland (2002), *Proof theory in the abstract*, Annals of Pure and Applied Logic, 114(1):43 - 78

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Our contributions

- Given a fibration p, when is there a fibration p' such that $\mathfrak{Dial}(\mathsf{p}') \cong \mathsf{p}$?
- When such fibration p' exists, how is it done?

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration*, arXive

Background

Definition

Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a functor and $X \xrightarrow{f} Y$ an arrow in E. Let us call $A \xrightarrow{u:=\mathrm{p}(f)} B$ the arrow $\mathrm{p}(f)$ of B. We say that f is **Cartesian over u** if, for every morphism $Z \xrightarrow{g} Y$ in E such that $\mathrm{p}(g)$ factors through u, $\mathrm{p}(g) = uw$, there exists a unique $Z \xrightarrow{h} X$ of E such that $g = fh$ and $\mathrm{p}(h) = w$.

Definition

A **fibration** is a functor $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ such that, for every Y in E and every $I \xrightarrow{u} \mathrm{p}Y$, there exists a Cartesian arrow $X \xrightarrow{f} Y$ over u.

Jacobs (1999), *Categorical Logic and Type Theory*, Studies in Logic and the foundations of mathematics, 141

Background

Definition

We say a fibration $p\colon E \longrightarrow B$ over a category B with finite products has **simple coproducts** when the weakening functors π^* have left adjoints $\coprod_\pi$ satisfying the *Beck-Chevalley Condition* (abbreviated as BCC).

Dually, we say that a fibration $p\colon E \longrightarrow B$ has **simple products** when the weakening functors π^* have right adjoints $\prod_\pi$ satisfying BCC.

Quantifier-free objects

The logical intuition behind the next definition is that an element α is *quantifier-free* if it satisfies the following universal property: if there is a proof π of a statement $\exists i\ \beta(i)$ assuming α, then there exists a *witness* t, which depends on the proof π, together with a proof of $\beta(t)$. Moreover, we require that this holds for every re-indexing $\alpha(f)$ because in logic quantifier-free propositions are stable under substitution.

Quantifier-free objects

Definition

Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a fibration with simple coproducts. An object α of the fibre E_I is said to be $\coprod$**-quantifier-free** if it enjoys the following universal property: for every pair of arrows

$$A \times B \xrightarrow{\pi_A} A \xrightarrow{f} I$$

and every vertical arrow:

$$f^*\alpha \xrightarrow{h} \coprod_{\pi_A} \beta$$

of E_A, where β is an object of the fibre $\mathrm{E}_{A\times B}$, there exist a unique arrow $A \xrightarrow{g} B$ of B and a unique vertical arrow $f^*\alpha \xrightarrow{\overline{h}} \langle 1_A, g\rangle^*\beta$ of E_A such that:

$$h = \left(\; f^*\alpha \xrightarrow{\overline{h}} \langle 1_A, g\rangle^*\beta \xrightarrow{\langle 1_A,g\rangle^*\eta_\beta} \langle 1_A, g\rangle^*(\pi_A^*\coprod_{\pi_A}\beta) = \coprod_{\pi_A}\beta \; \right)$$

Quantifier-free objects

Definition

We say that a fibration with simple coproducts $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ has **enough $\coprod$-quantifier-free objects** if, for every object I of B and for every element $\alpha \in \mathrm{E}_I$, there exist an object A and a $\coprod$-quantifier-free object β in $\mathrm{E}_{I\times A}$ such that $\alpha \cong \coprod_{\pi_I} \beta$.

Quantifier-free objects

Definition

Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a fibration with simple products. An object α of the fibre E_I is said to be $\prod$**-quantifier-free** if it enjoys the following universal property: for every arrow f and every projection π_A in B as follows:

$$A \times B \xrightarrow{\pi_A} A \xrightarrow{f} I$$

and every vertical arrow:

$$\prod_{\pi_A} \beta \xrightarrow{h} f^*\alpha$$

of E_A, where β is an object of the fibre $\mathrm{E}_{A\times B}$, there exist a unique arrow $A \xrightarrow{g} B$ of B and a unique vertical arrow $\langle 1_A, g\rangle^*\beta \xrightarrow{\overline{h}} f^*\alpha$ of E_A such that:

$$h = \left(\prod_{\pi_A} \beta = \langle 1_A, g\rangle^* (\pi_A^* \prod_{\pi_A} \beta) \xrightarrow{\langle 1_A, g\rangle^* \varepsilon_\beta} \langle 1_A, g\rangle^*\beta \xrightarrow{\overline{h}} f^*\alpha \right)$$

Quantifier-free objects

Definition

We say that a fibration with simple products $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ has **enough-$\prod$-quantifier-free objects** if, for every object I of B and for every element $\alpha \in \mathsf{E}_I$, there exist an object A and a $\prod$-quantifier-free object β in $\mathsf{E}_{I\times A}$ such that $\alpha \cong \prod_{\pi_I}(\beta)$.

Skolem fibration

Definition

A fibration $p\colon E \longrightarrow B$ is called a **Skolem fibration** if:

- its base category B is cartesian closed;
- the fibration p has simple products and simple coproducts;
- the fibration p has enough $\coprod$-quantifier-free objects.
- $\coprod$-quantifier-free objects are stable under simple products, i.e. if $\alpha \in E_I$ is a $\coprod$-quantifier-free object, then $\prod_\pi(\alpha)$ is a $\coprod$-quantifier-free object for every projection π from I.

Skolem fibration

Theorem (Skolemization)

Every Skolem fibration p *validates the principle:*

$$\forall x \exists y \alpha(i,x,y) \cong \exists f \forall x \alpha(i,x,fx).$$

Gödel fibration

Definition

A Skolem fibration $p \colon E \longrightarrow B$ is called a **Gödel** fibration if the sub-fibration $\bar{p} \colon \bar{E} \longrightarrow B$, whose elements are $\prod$-quantifier-free objects, has enough $\prod$-quantifier-free objects.

Gödel fibration

Theorem (Prenex normal form)

In a Gödel fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$, *for every element* α *of a fibre* E_I *there exists an element* β *such that*

$$\alpha(i) \cong \exists x \forall y \beta(x, y, i)$$

and β *is* $\prod$*-quantifier-free in the sub-fibration* $\bar{\mathsf{p}}$ *of* $\coprod$*-quantifier-free objects of* p.

The Dialectica fibration

Dialectica construction. Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a fibration, whose base category is cartesian closed. Define a category $\mathfrak{Dial}(\mathrm{p})$ as follows:

- **objects** are quadruples (I, X, U, α) where I, X and U are objects of the base category B and $\alpha \in \mathrm{E}_{I\times X\times U}$ is an objects of the fibre of p over $I \times X \times U$;
- a **morphism** from (I, X, U, α) to (J, Y, V, β) is a quadruple (f, f_0, f_1, ϕ) where
  1. $I \xrightarrow{f} J$ is a morphism in B;
  2. $I \times X \xrightarrow{f_0} Y$ is a morphism in B;
  3. $I \times X \times V \xrightarrow{f_1} U$ is a morphism in B;
  4. $\alpha(i, x, f_1(i, x, v)) \xrightarrow{\phi} \beta(f(i), f_0(i, x), v)$ is an arrow in the fibre over $I \times X \times V$.

Skolemization

Theorem

When the base category B *of a fibration* p *is cartesian closed, the fibration* $\mathfrak{Dial}(\mathsf{p})$ *satisfies the principle*

$$\forall x \exists y \alpha(i,x,y) \cong \exists f \forall x \alpha(i,x,fx)$$

for every α.

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Our contribution

Theorem

Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ *be a fibration with simple products, coproducts and such that* B *is cartesian closed. Then there exists a fibration* p' *such that* $\mathfrak{Dial}(\mathrm{p}') \cong \mathrm{p}$ *if and only if* p *is a Gödel fibration.*

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration*, arXive

Sketch of the proof

The original Dialectica construction $\mathfrak{Dial}$ can be seen as the composition of two free constructions $\mathfrak{Sum}$ and $\mathfrak{Prod}$, which are the simple sum (or co-product) and the simple product completions, respectively.

Lemma

There is an isomorphism of fibrations, natural in p*:*

$$\mathfrak{Dial}(\mathrm{p}) \cong \mathfrak{Sum}(\mathfrak{Prod}(\mathrm{p})).$$

These completions are fully dual, in particular $\mathfrak{Prod}(\mathrm{p}) \cong \mathfrak{Sum}(\mathrm{p}^{\mathrm{op}})^{\mathrm{op}}$, so we only need to study one and can then deduce results for the other construction.

Hofstra (2011), *The dialectica monad and its cousins*, Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai, 53:107-139

Sketch of the proof

Theorem

A fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *with simple coproducts is an instance of simple coproduct completion if and only if it has enough* $\coprod$*-quantifier-free objects. Moreover, in this case* $\mathsf{p} \cong \mathfrak{Sum}(\mathsf{p}')$ *where* p' *is the subfibration of* $\coprod$*-free-quantifiers objects of* p*.*

Theorem

A fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *with simple products is an instance of simple product completion if and only if it has enough-* $\prod$*-quantifier-free objects. Moreover, in this case* $\mathsf{p} \cong \mathfrak{Prod}(\mathsf{p}'')$ *where* p'' *is the subfibration of* $\prod$*-free-quantifiers objects of* p*.*